The maximum possible error that can occur during a measurement on a phase bridge is therefore equal to the sum of (2) and (4), and is always a function of the deviation from nominal impedance at the terminals of the device under test. However, it is very unlikely that the line lengths between discontinuities would be arranged so that the error would achieve the maximum possible value during any measurement.

Unequal power division between the two bridge arms will introduce no direct error. However, if the inequality is large enough to decrease appreciably the sharpness of the output null, error may be introduced because of increased uncertainty in the exact setting of the line stretchers.

Conclusion

Both phase shifters have the advantage of small physical size, potential rugged construction and linear relation between phase shift and mechanical motion. The low driving torque necessary to accomplish phase shift is an attractive feature for servo-control applications.

The helical phase shifter has a bandwidth advantage over the coupler type but has a greater insertion loss. The bandwidth of the helix phase shifter extends from UHF up through C band, whereas the bandwidth of the coupler type is limited to less than an octave. However, the linearity of the coupler type is somewhat greater than in the helical phase shifter. Both designs are intended for low-power receiver types of applications, and maximum power-handling capacity has not been determined.

For precise phase measurements, bridge methods are recommended rather than slotted-line techniques. The chief source of measurement error is caused by interactions between impedance deviations at the terminals of the phase shifter and deviations at the terminals of the testing facility. The maximum possible phase errors that can occur as a result of the deviations can be calculated by the methods that have been given.

A Dielectric Resonator Method of Measuring Inductive Capacities in the Millimeter Range*

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Summary—**A novel technique for the measurement of dielectric and magnetic properties of a homogeneous isotropic medium in the range of approximately 3 to 100 kmc is described. An accuracy of ∓0.1 per cent is possible in the determination of permittivity or permeability in those cases where the loss tangent is sufficiently small. The measuring structure is a resonator made up of a right circular cylindrical dielectric rod placed between two parallel conducting plates. For measurement of permittivity two or more resonant TE_{onl} mode frequencies are determined whereas for the measurement of permeability two or more resonant TM_{onl} mode frequencies are determined. The dielectric or magnetic properties are computed from the resonance frequencies, structure dimensions, and unloaded Q. Since the loss tangent is inversely proportional to the unloaded Q of the structure, the precision to which Q is measured determines the accuracy of the loss tangent.**

I. Introduction

Most of the methods for the measurement of dielectric and magnetic properties of materials at microwave frequencies fall into the following categories: perturbation techniques, optical methods, transmission line methods, or exact resonance methods. The perturbation techniques have notably been used in ferrite measurements where a small piece of the material to be measured, either in the form of a disk or a sphere, is placed in a metallic resonant cavity operating in a known mode and the shift in resonant frequency and the Q of the structure is noted.[1-4] Optical techniques at microwave frequencies[5] are inherently suited for measurements below a wavelength of one centimeter, but

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[1] W. Von Aulock and J. H. Rowen, "Measurement of dielectric and magnetic properties of ferromagnetic materials at microwave frequencies," *Bell Sys. Tech. J.*, vol. 36, pp. 427–448; March, 1957.

[2] E. G. Spencer, R. C. LeCraw, and F. Reggia, "Measurement of microwave dielectric constants and tensor permeabilities of ferrite spheres," Proc. IRE, vol. 44, pp. 790–800; June, 1956.

[3] J. O. Artman and P. E. Tannenwald, "Measurement of permeability tensor in ferrites," *Phys. Rev.*, vol. 91, pp. 1014–1015; August 15, 1953.

[4] J. O. Artman and P. E. Tannenwald, "Microwave susceptibility measurements in ferrites," *J. Appl. Phys.*, vol. 26, pp. 1124–0000; September, 1955.

[5] T. E. Talpey, "Optical methods for the measurement of complex dielectric and magnetic constants at centimeter and millimeter wavelengths," IRE Trans. on Microwave Theory and Techniques, vol. MTT-2, pp. 1–13; September, 1954.

their only drawback is that a relatively large amount of the material is needed. Finally, transmission line techniques have been applied widely[6,7] but they have the very serious disadvantage of the very small waveguide size used below 4 mm, which gives rise to practical difficulties. All the above methods have a rather limited accuracy of the order of ∓ 1 per cent in the values of permittivity and permeability obtained. On the other hand an exact resonance method has recently been proposed by Karpova[8] which gives an accuracy of ∓ 0.1 per cent. In this method a circular disk of the material to be measured is inserted in the gap of a reentrant cavity of known dimensions, Fig. 1, and the resonant frequency and Q of the structure are measured from which the dielectric constant and loss tangent are obtained. This is a high accuracy method but the physical size of the resonant structure for the low millimeter range might pose a problem unless a high order mode cavity is used.[9]

In order to circumvent the problem of physical size and still maintain a high accuracy, an open boundary resonant structure was used in the form of a dielectric rod, of the material to be measured, placed between two mathematically infinite conducting plates.[10]

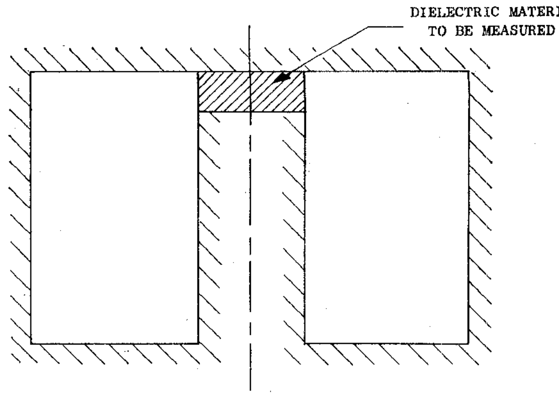

Fig. 1—A reentrant cavity structure used by Karpova[8] to measure dielectric properties.

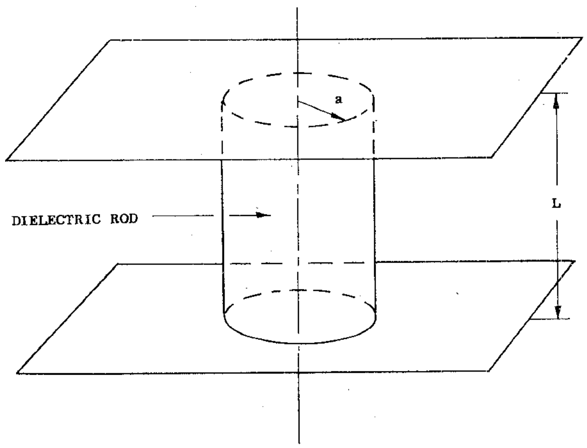

Fig. 2—Dielectric rod placed between two infinite parallel perfectly conducting plates.

II. General Analysis

A. Measurement of Permittivity

Let it be assumed for the present that the permeability of the material of interest is equal to that of free space, and it is required to measure the permittivity. Consider now a circular cylindrical rod of relative dielectric constant κ, permeability μ_0, length L, and radius a placed between two large perfectly conducting plates (Fig. 2). If the dielectric material is isotropic then the characteristic equation for this resonant structure operating in the TE_{onl} mode is (see Appendix I)

$$\alpha \frac{J_0(\alpha)}{J_1(\alpha)} = -\beta \frac{K_0(\beta)}{K_1(\beta)} \tag{1}$$

where $J_0(\alpha)$ and $J_1(\alpha)$ are the Bessel functions of the first kind of orders zero and one, respectively, $K_0(\beta)$ and $K_1(\beta)$ are the modified Bessel functions of the second kind of orders zero and one, respectively. The parameters α and β depend on the geometry, the resonant wavelength, and dielectric properties; thus

$$\alpha = \frac{2\pi a}{\lambda_0}\left[\kappa_1 - \left(\frac{c}{v_p}\right)^2\right]^{1/2} \tag{2}$$

$$\beta = \frac{2\pi a}{\lambda_0}\left[\left(\frac{c}{v_p}\right)^2 - 1\right]^{1/2} \tag{3}$$

c being the velocity of light and v_p the phase velocity in the structure so that

$$\frac{c}{v_p} = \left(\frac{l\lambda_0}{2L}\right) \tag{4}$$

where l is the number of longitudinal variations of the field along the axis, and λ_0 is the free space resonant wavelength.

It is seen that the characteristic equation is a transcendental equation and hence a graphical solution is necessary. The resulting mode charts are given in Figs. 3 and 4 where to each value of β there corresponds an infinite sequence $\{\alpha_n\}$ which solves (1); the definition of the subscript n is now self-evident.

[6] C. Montgomery, "Technique of Microwave Measurements," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 11; 1947.

[7] A. R. Von Hippel, "Dielectric Materials and Applications," John Wiley & Sons, Inc., New York, N. Y.; 1956.

[8] O. V. Karpova, "On the absolute measurement of solid dielectric parameters by means of a π resonator," *Soviet Phys.*, vol. 1 (Solid-State), pp. 220–228; February, 1959.

[9] H. E. Bussey and L. A. Steinert, "Exact solution for a gyromagnetic sample and measurements on a ferrite," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 72–76; January, 1958.

[10] This work has similar aspects to the exhaustive and excellent work done at Northwestern University by R. E. Beam, *et al.*, whose results are included in the final report, "Investigations on Multi-Mode Propagation in Waveguides and Microwave Optics"; November, 1950. Also, a similar idea for a slow-wave resonant structure was independently proposed by Prof. R. Pantell and the details carried out by Dr. R. Becker in his Doctoral dissertation, "The Dielectric Tube Resonator." See also, R. H. Pantell, P. D. Coleman, and R. C. Becker, "Dielectric slow-wave structures for the generation of power at millimeter and submillimeter wavelengths," IRE Trans. on Electron Devices, vol. ED-5, pp. 167–173; July, 1958.

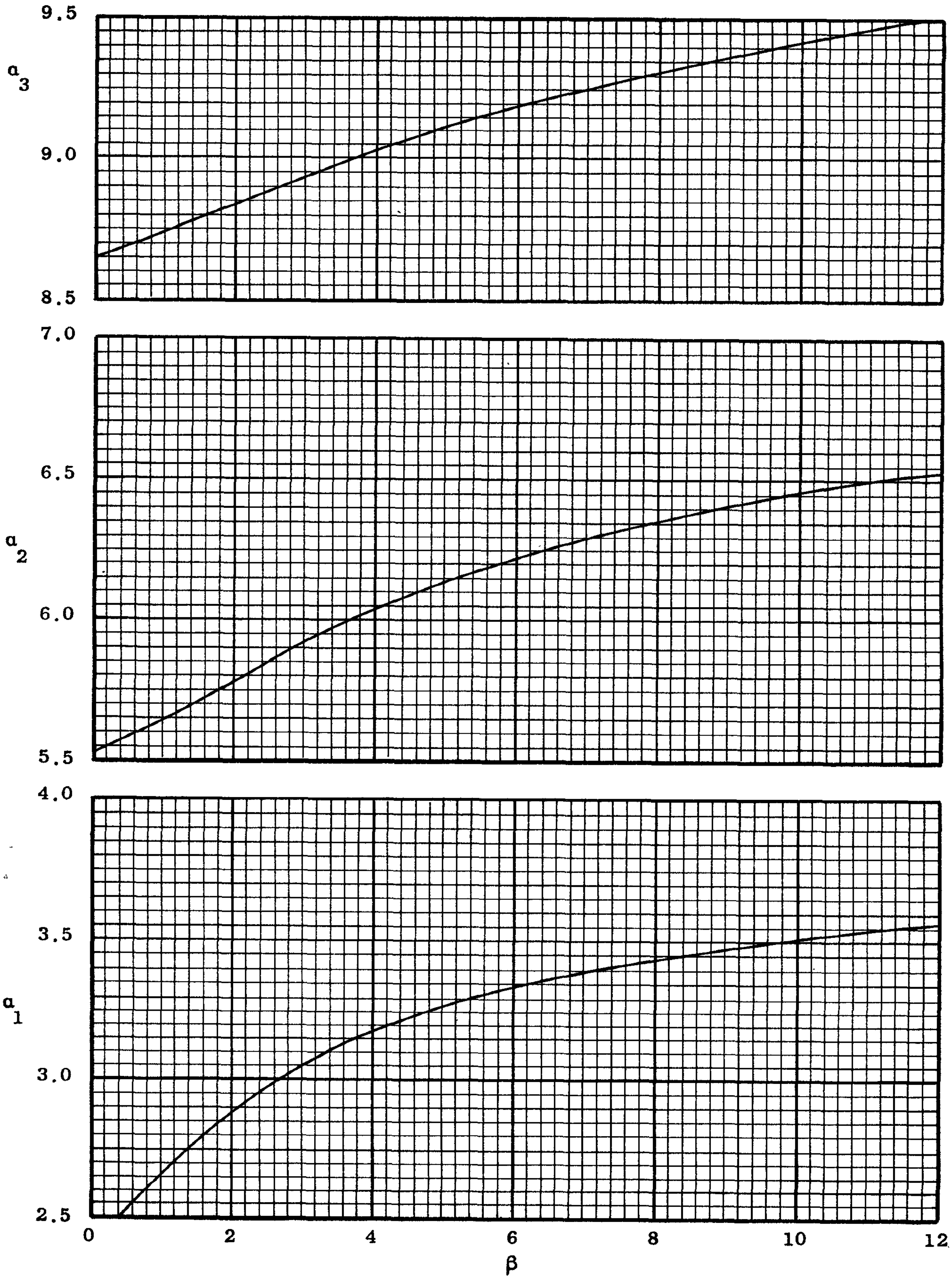

Fig. 3—The solution of the characteristic equation for TE_{onl} modes for $n=1$ up to 3.

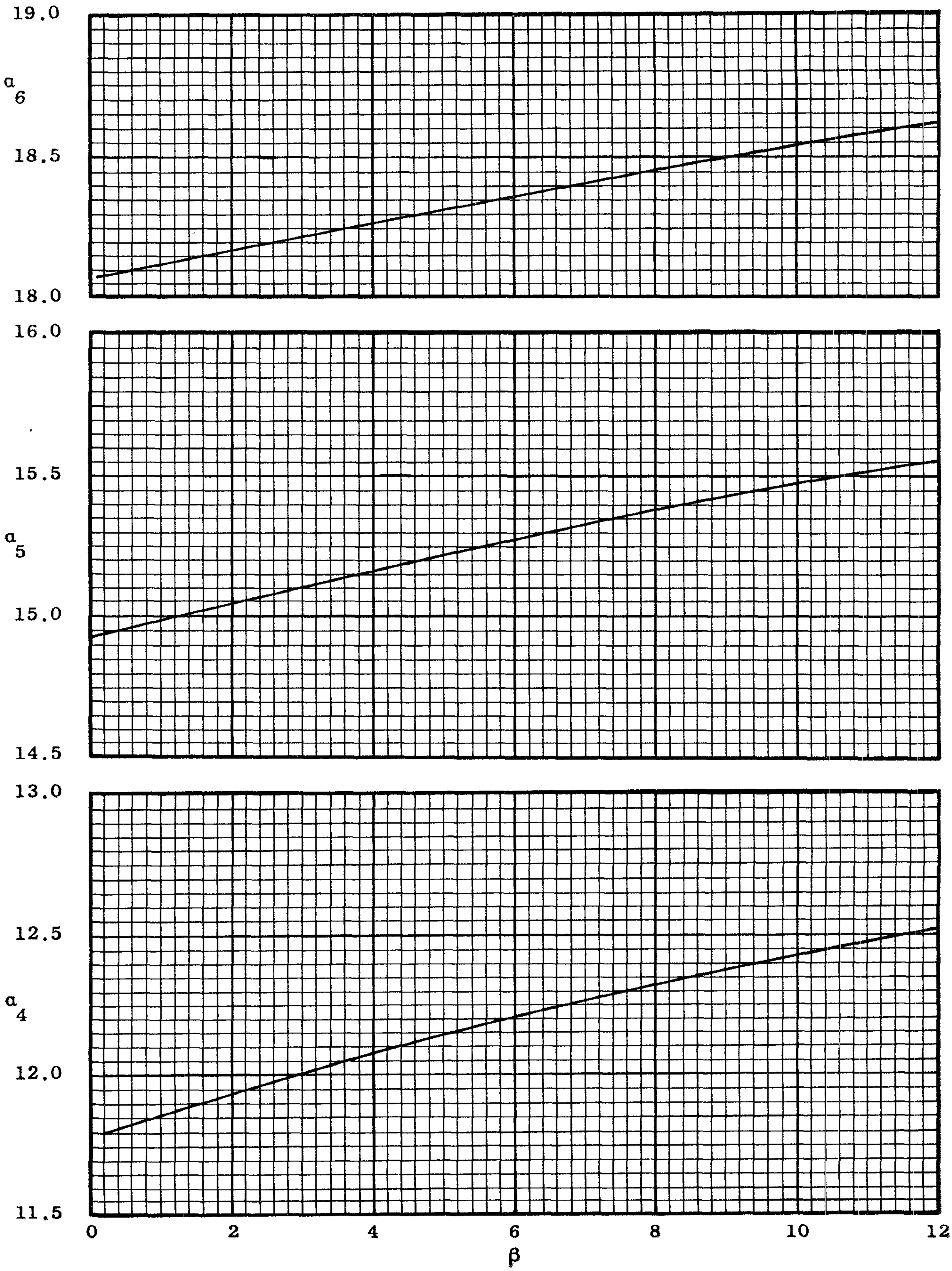

Fig. 4—The solution of the characteristic equation for TE_{onl} modes for $n=4$ up to 6.

Noteworthy in (1) is the fact that the characteristic equation and the resulting mode charts are "universal" as far as the permittivity is concerned. This makes possible the use of the mode charts for practically all values of relative dielectric constants conceivable. Consider now the problem of determining the relative dielectric constant of a structure of known dimensions resonating in a TE_{onl} mode with a resonant wavelength λ_0. Eq. (4) is substituted in (3) which gives

$$\beta_l = \frac{2\pi a}{\lambda_0}\left[\left(\frac{l\lambda_0}{2L}\right)^2 - 1\right]^{1/2}. \tag{3a}$$

But to β_l there corresponds an α_n which can be obtained from the mode chart of the characteristic equation. Once the appropriate value of α_n is obtained the relative dielectric constant can be immediately calculated from (2), assuming that the mode indexes n and l are known,

$$\kappa_1 = \left(\frac{\alpha_n\lambda_0}{2\pi a}\right)^2 + \left(\frac{l\lambda_0}{2L}\right)^2. \tag{5}$$

However, in practice the most difficult aspect of the problem is the determination of the mode indexes. In the first place the modes of interest are restricted to the TE_{onl} which form a small portion of all the possible modes of operation, hence the method of excitation has to be such as to excite TE_{onl} modes exclusively. Unfortunately, such a highly selective method of excitation is rather difficult to obtain, thus it should be expected that spurious resonances other than the TE_{onl} modes will be present. In the second place, even after the excitation of the TE_{onl} mode is accomplished there remains the determination of n and l. The most obvious method of determining l would be by actually counting the longitudinal variations of the field by a perturbation technique. However, at wavelengths below 8 mm and for high relative dielectric constants, say about 1000, the spacing of the nulls along the cavity is of the order of 0.1 mm which is not all too easy to measure by a perturbation method; another method seems to be necessary.

If the dielectric constant and cavity dimensions give rise to an α_n and a β_l which fall in a region of the mode chart where there is enough discrimination then a very simple procedure is possible. Suppose that a structure with certain dimensions has a resonant frequency λ_0; since the particular integral value of l is not known, a table of β_l, obtainable from (3), is arranged for all values of l ranging from one up to any reasonable value. Now for each value of β_l several values of α_n are obtained from the mode chart; next, for each pair of values (α_n, β_l) a dielectric constant is obtained from (5). Another resonant cavity with different dimensions is designed and the same procedure is repeated; the value of dielectric constant common to both sets of values is the correct value. To eliminate any ambiguity a third cavity is designed so as to resonate at a certain frequency by using the obtained value of κ, and it is observed whether the actual resonant frequency is close to the designed value to within certain tolerances. If the signal source has a wide enough frequency range and if the dielectric properties remain nearly constant throughout this range then it might be possible to use the same cavity and excite two different TE_{onl} modes. An illustrative example is given for teflon in Table I, which gives a value of 2.044 for the relative dielectric constant at room temperature and a frequency of 35 kmc.

TABLE I

PROCEDURE FOLLOWED IN OBTAINING THE RELATIVE DIELECTRIC CONSTANT OF TEFLON AT 25°C. CAVITY DIMENSIONS ARE L=18.81 MM, a=14.44 MM; κ=2.044.

	κ			λ_0 mm
	l=5	l=6	l=7	
n=1	1.393	1.979	2.654	
=2	1.638	2.237	2.925	
=3	**2.044**	2.656	3.353	8.564
=4	2.621	3.248		
=5	3.371			
n=1	1.443	**2.044**	2.746	
=2	1.698	2.314	3.026	
=3	2.120	2.749		8.713
=4	2.718	3.355		
=5	3.493			

B. Measurement of the Loss Tangent

Once the mode of operation and the dielectric constant are determined, it is a simple matter to derive the loss tangent of the dielectric material from a measured value of the unloaded Q of the structure. The unloaded Q of the cavity is defined as 2π times the ratio of the maximum energy stored to the energy lost in one cycle. The energy stored consists of energy within and outside the dielectric rod whereas energy losses consist of copper losses in the end walls and dielectric losses in the rod itself. An expression for the loss tangent of the material used can be derived and gives (see Appendix II)

$$\tan\delta = \frac{A}{Q_0} - B \tag{6}$$

where

$$A = 1 + \frac{J_1^2(\alpha_n)}{\kappa K_1^2(\beta_l)}\left[\frac{K_0(\beta_l)K_2(\beta_l) - K_1^2(\beta_l)}{J_1^2(\alpha_n) - J_0(\alpha_n)J_2(\alpha_n)}\right] \tag{7}$$

and

$$B = \frac{l^2R_s}{2\pi f^3\mu^2\kappa\epsilon_0L^3}\left\{1 + \frac{J_1^2(\alpha_n)}{K_1^2(\beta_l)}\left[\frac{K_0(\beta_l)K_2(\beta_l) - K_1^2(\beta_l)}{J_1^2(\alpha_n) - J_0(\alpha_n)J_2(\alpha_n)}\right]\right. \tag{8}$$

where the terms have the usual definition, α_n and β_l being the solutions of the characteristic equation (1) for the particular mode for which the Q_0 of the structure has been determined, and R_s is the surface resistance of the end walls, equal to $(\pi f\mu/\sigma)^{1/2}$, σ being the conductivity. It is seen from (6) that when $\kappa \gg 1$, as for ferroelectric materials, and end-wall losses are neglected, then the loss tangent is approximately

$$\tan\delta \approx \frac{1}{Q_0}\cdot$$

In order to facilitate computations a graphical plot of the factors involved in (7) and (8) has been prepared. Thus A and B can be written as

$$A = 1 + \frac{1}{\kappa} F(\alpha)G(\beta) \tag{7a}$$

$$B = \frac{l^2 R_s}{2\pi f^3 \mu^2 \kappa \epsilon_0 L^3}\left[1 + F(\alpha)G(\beta)\right] \tag{8a}$$

where the definitions of $F(\alpha)$ and $G(\beta)$ are self-evident. Plots of $F(\alpha)$ vs α and $G(\beta)$ vs β are given in Figs. 5 and 6.

Finally, since this is an inherently high accuracy method it would be of interest to study the effect of losses on the resonant frequency. For this purpose the expressions derived by Slater[11] for the effect of losses can be used. If ω_0 is the resonant frequency of the lossless structure, and $\tan\delta$ is the dielectric loss coefficient of the medium, then the resonant frequency of the structure when dielectric losses are accounted for is[12]

$$\omega = \omega_0\left[1 - \left(\frac{\tan\delta}{2}\right)^2\right]^{1/2}. \tag{9}$$

Now, for $(\frac{1}{2}\tan\delta)^2 \ll 1$, (9) can be expanded and gives

$$\omega_0 \approx \omega\left[1 + \tfrac{1}{8}(\tan\delta)^2\right]. \tag{9a}$$

It is seen that even for a $\tan\delta = 0.1$ the resulting shift in the resonant frequency is about 0.1 per cent. In addition, effects of copper losses on the resonant frequency can be safely neglected.

C. Measurement of Permeability

The same technique can be applied for the measurement of permeability as that indicated above except for one difference: in order to get a characteristic equation which is "universal" for all permeabilities it would be necessary to use the TM_{onl} modes which give rise to a characteristic equation of the form

$$\frac{\alpha J_0(\alpha)}{J_1(\alpha)} = -\kappa\beta\frac{K_0(\beta)}{K_1(\beta)} \tag{10}$$

where the relative dielectric constant κ has been retained in case the magnetic material has a permittivity different from that of free space, and the independent variables α and β are now given by

$$\alpha = \frac{2\pi a}{\lambda_0}\left[\mu_r\kappa - \left(\frac{l\lambda_0}{2L}\right)^2\right]^{1/2} \tag{11}$$

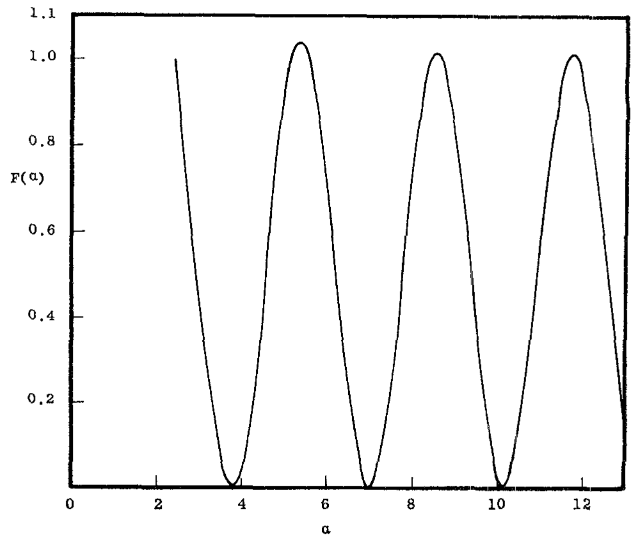

Fig. 5—$F(\alpha)$ vs α.

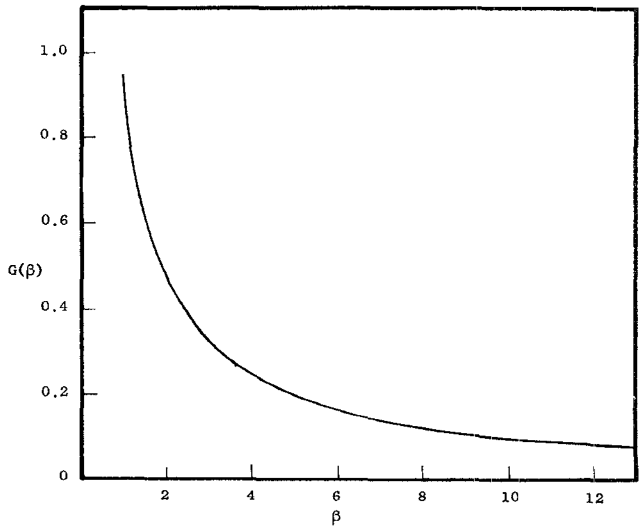

Fig. 6—$G(\beta)$ vs β.

$$\beta = \frac{2\pi a}{\lambda_0}\left[\left(\frac{l\lambda_0}{2L}\right)^2 - 1\right]^{1/2}, \tag{12}$$

μ_r being the relative permeability of the material. It is obvious from (10) that should the magnetic material have a relative dielectric constant equal to one then (10) becomes identical to (1), and the mode charts already obtained would be applicable; otherwise a new set of mode charts must be derived for each value of relative dielectric constant κ. Once this is accomplished the procedure follows through in an identical fashion to that for the determination of permittivity, and further exposition is not warranted.

D. Experimental Details

Consider next experimental aspects of the measurement of permittivity. It has already been pointed out that the modes relevant to permittivity measurements are the TE_{onl} modes. In practice it has been found that

[11] J. Slater, "Microwave Electronics," D. Van Nostrand Co., Inc., New York, N. Y.; 1950.

[12] The results obtained by Slater are still valid in this case in spite of the lack of orthogonality of the eigenfunctions in the finite interval $0 < r < a$.

an iris coupling in the narrow side of a guide operating in the dominant TE_{01} mode gives satisfactory results. The resonant structure in this case behaves as a reaction type cavity; details of the structure are shown in Fig. 7. It was found to be almost imperative to incorporate some tuning mechanism in the resonant structure, *e.g.*, moving one end wall, in order to detect the structure resonance on, for example, the saw-tooth modulated output of the klystron. This is due to the fact that most dielectric materials have rather large losses at microwave frequencies and hence give rise to low Q structures. In addition, the coupling may not be strong enough to give a noticeable depression on the bell-shaped output of the klystron as seen on an oscilloscope. Thus tuning the structure by moving one end wall was found to be helpful in locating the resonances.[13] The cavity can be placed in the microwave circuit in any of the conventional ways, *e.g.*, either on line or by the use of a magic-T bridge.

The Q of the resonant cavity is measured by any of several possible ways, the VSWR method[14] being used in this particular case. The results for several materials have been included in Table II.

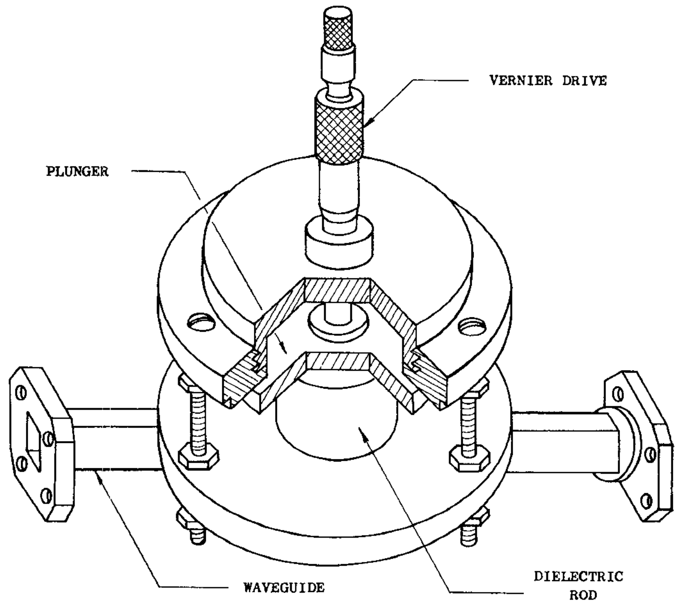

Fig. 7—The dielectric rod cavity arrangement. A section of one end wall is driven by a vernier for tuning purposes.

TABLE II

Dielectric Constant and Loss Tangent of Several Materials at Room Temperature at a Frequency of 35 KMC

			Values Quoted by Other Authors			
Material	κ	tan δ	Author	κ	tan δ	Frequency (Kmc)
Teflon	2.044 $\mp$0.2 per cent	10×10^{-4} $\mp$10 per cent	Von Hippel[7] Handbook[15]	2.08 2.08	6×10^{-4} 6×10^{-4}	25 25
Polystyrene	2.543 $\mp$0.2 per cent	33×10^{-4}	Von Hippel[7] Karpova[6] Talpey[5]	2.54 2.495 2.52	12×10^{-4} 4.9×10^{-4} 92×10^{-4}	25 2 36
Lucite	2.580 $\mp$0.2 per cent	50×10^{-4}	Von Hippel[7] Handbook[15]	2.57 2.57	32×10^{-4} 32×10^{-4}	25 25

III. Conclusion

By measuring the resonant frequency of a dielectric rod between two large parallel conducting plates it was possible to measure the dielectric constant of teflon, polystyrene, and lucite to an accuracy better than $\mp$0.2 per cent. The accuracy was limited by the probable error in the wave meter and errors in taking the dimensions of the samples. The error is much less for low loss materials, *i.e.*, high Q resonant structures, since the absorption curve of the resonant structure is more sharply displayed on the bell-shaped output of a saw-tooth modulated klystron. Errors in the mode chart can be theoretically reduced to an arbitrarily small value through more accurate numerical computations. The loss tangent has errors in it which are inherent in those involved in measurement of Q. It is quite possible to use a frequency stabilized circuit such as that used by Von Aulock[1] to improve on the accuracy of Q measurements. It should be pointed out, however, that this method would lose its main advantage—accuracy—in those cases where both the dielectric and magnetic properties are simultaneously unknown and an auxiliary technique is used to determine one or the other.

One of the interesting features of this scheme is its possibility in measurement of dielectric constant and loss tangent of nonlinear dielectric materials, *e.g.*, ferroelectrics. If a dielectric rod resonant structure is made of such a material and its resonance absorption curve

[13] Needless to say, when data are taken the end walls should be in direct contact with the cylindrical rod.

[14] E. L. Ginzton, "Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y.; 1957.

[15] "Reference Data for Radio Engineers," Federal Telephone and Radio Co., New York, N. Y., 3rd ed.; 1949.

displayed on the bell-shaped output of a klystron, then it would be possible to study the effect of the strength of excitation on the dielectric properties of the nonlinear material by noting the frequency shift and distortion of the resonance curve. This, added to the fact that this scheme is inherently suited to the study of the effect of dc electric bias on the dielectric properties since dc voltage can easily be applied between the two end walls, yielding information concerning the incremental permittivity of the material. Naturally, this is not without its own share of problems; for instance, for ferroelectric materials the size of the cavity has to be small if it is to operate in a reasonably low order mode. Also, with the high dielectric constant of ferroelectric materials there is the problem of impedance matching, where a low impedance matching device is needed. Nevertheless, it is believed that such difficulties can be overcome.

Appendix I

The Characteristic Equation for TE_{onl} Modes

The characteristic equation for the TE_{onl} modes in a dielectric rod resonant structure will be obtained. The usual vector $\boldsymbol{L}$, $\boldsymbol{M}$, and $\boldsymbol{N}$ solutions of the vector Helmholtz equation are convenient and hence will be used. Since TE modes are of interest let

$$\boldsymbol{E}(r) = \boldsymbol{M}(r) = \nabla \times [\chi\psi(r)] \tag{13}$$

which in cylindrical coordinates gives

$$\boldsymbol{E}(r) = \nabla\psi(r) \times \boldsymbol{i}_z \tag{14}$$

where the scalar function $\psi(r)$ satisfies the scalar Helmholtz equation

$$\nabla^2\psi(r) + \omega^2\mu\epsilon\psi(r) = 0. \tag{15}$$

Eq. (15) is to be solved in the two regions I and II, *i.e.*, inside the rod and outside the rod, respectively, such as to satisfy the appropriate boundary conditions.

Region I

If there is no ϕ variation then a solution of the above equation will be

$$\psi_1(r) = A_1J_0(k_1r) \sin \gamma z \tag{16}$$

where

$$k_1^2 = \omega^2\mu\epsilon_1 - \gamma^2, \tag{17}$$

and

$$\boldsymbol{E}_1(r) = A_1\nabla[J_0(k_1r) \sin \gamma z] \times \boldsymbol{i}_z$$

$$\therefore \boldsymbol{E}_1(r) = \boldsymbol{i}_\phi A_{1k}J_1(k_1r) \sin \gamma z. \tag{18}$$

It is seen that $\boldsymbol{E}_1(r)$ satisfies the boundary conditions at $z=0$ and L if

$$\gamma = \frac{l\pi}{L} \qquad l = 1, 2, 3, \cdots. \tag{19}$$

Next, from Maxwell's first equation we have

$$\boldsymbol{H}_1(r) = \frac{j}{\omega\mu} \nabla \times \boldsymbol{E}_1(r)$$

$$\boldsymbol{H}_1(r) = \frac{jA_1k_1}{\omega\mu}[-\boldsymbol{i}_r\gamma J_1(k_1r) \cos \gamma r + \boldsymbol{i}_zk_1J_0(k_1r) \sin \gamma z]. \tag{20}$$

Region II

The scalar potential $\psi_2(r)$ is a solution of

$$\nabla^2\psi_2(r) + \omega^2\mu\epsilon_0\psi_2(r) = 0 \tag{21}$$

hence

$$\psi_2(r) = A_2K_0(k_2r) \sin \gamma z$$

where

$$k_2^2 = \gamma^2 - \omega^2\mu\epsilon_0. \tag{22}$$

It is necessary here to use the modified Bessel functions since the modes are nonradiating. Thus from (14) and (22) we obtain

$$\boldsymbol{E}_2(r) = \boldsymbol{i}_\phi A_2k_2K_1(k_2r) \sin \gamma z \tag{23}$$

and finally

$$\boldsymbol{H}_2(r) = \frac{-jA_2k_2}{\omega\mu}[\boldsymbol{i}_r\gamma K_1(k_2r) \cos \gamma z + \boldsymbol{i}_zk_2K_0(k_2r) \sin \gamma z]. \tag{24}$$

The boundary condition of continuity of the tangential components of the fields at $r=a$ gives rise to the characteristic equation

$$\alpha \frac{J_0(\alpha)}{J_1(\alpha)} = -\beta\frac{K_0(\beta)}{K_1(\beta)} \tag{25}$$

where

$$\alpha = ak_1 \quad \text{and} \quad \beta = ak_2.$$

Appendix II

The Q of the Structure for TE_{onl} Modes

The unloaded Q of any resonant structure is defined as 2π times the ratio of the maximum stored energy to the energy lost in one cycle. The energy storage takes place partly in the dielectric rod and partly outside the rod, similarly the energy losses are due to losses in the end walls and dielectric losses in the cavity. Thus

$$Q_0 = \frac{\omega(U_{\text{cav}} + U_{\text{sp}})}{W_{\text{walls}} + W_{\text{cav}}}\cdot \tag{26}$$

Now, the stored energy is

$$U_{\text{cav}} = \int_{\text{cav}} \frac{\epsilon_1}{2} \mid \boldsymbol{E}_1(r) \mid^2 dr \tag{27}$$

$$U_{\text{sp}} = \int_{\text{sp}} \frac{\epsilon_0}{2} \mid \boldsymbol{E}_2(r) \mid^2 dr \tag{28}$$

whereas the energy losses are

$$W_{\text{walls}} = R_s \int_{\text{walls}} | H(r) \times i_z |^2 dS \tag{29}$$

where R_s is the surface resistance of the walls and is equal to $R_s = (\pi f \mu / \sigma)^{1/2}$, f being the frequency, μ the permeability, and σ the conductivity. The energy losses in the cavity are

$$W_{\text{cav}} = \frac{\omega \epsilon_1 \tan \delta}{2} \int_{\text{cav}} | E_1(r) |^2 dr. \tag{30}$$

Eqs. (27)–(30) can be evaluated for the TE_{onl} mode field components given in Appendix I and the result manipulated to give tan δ, the dielectric loss tangent, in terms of known parameters. The result is

$$\tan \delta = \frac{1}{Q_0}\left[1 + \frac{1}{\kappa} F(\alpha) G(\beta)\right] - \frac{l^2 R_s}{2\pi f^3 \mu^2 \kappa \epsilon_0 L^3}\left[1 + F(\alpha) G(\beta)\right] \tag{31}$$

where

$$F(\alpha) = J_1^2(\alpha_n)/[J_1^2(\alpha_n) - J_0(\alpha_n) J_2(\alpha_n)]. \tag{32}$$

and

$$G(\beta) = [K_0(\beta_l) K_2(\beta_l) - K_1^2(\beta_l)]/K_1^2(\beta_l). \tag{33}$$

Graphical plots of $F(\alpha)$ vs α and $G(\beta)$ vs β are given in Figs. 5 and 6, respectively.

Summary of Measurement Techniques of Parametric Amplifier and Mixer Noise Figure*

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Summary—Expressions are derived for the noise factor of a frequency mixing circuit under two different operating conditions: 1) single-sideband operation with input only in a band of frequencies at ω_1; and 2) double-sideband radiometer operation with incoherent inputs in the bands both at frequency ω_1 and at $\omega_2 = \omega_3 - \omega_1$. In both cases, the output is taken only at ω_1.

It is shown that the noise figure for radiometer double-sideband operation is not always 3 db less than for single-sideband operation. It is also shown that it is possible to obtain an output signal-to-noise ratio which is greater than the input signal-to-noise ratio for coherent double-sideband operation.

Methods are analyzed for measuring the effective noise temperature of this circuit by using a broad-band noise source.

I. Introduction

THE AUTHOR of this paper has recently had occasion to attempt measurements of the noise figure of a quasi-degenerate parametric microwave amplifier.[1] During the course of this experiment, it was quite surprising to find that the measured values of noise figure were less than 3 db, although a simple theory[2,3] (which was believed to be fairly well understood) predicted that the noise figure of this device should always be greater than 3 db. It was soon realized that this apparent anomaly had come about because the method of measurement (with a broad-band noise tube) gave the radiometer double-sideband rather than the single-sideband figure, whereas the theoretical calculations were for single-sideband operation only.

Further reflection on this problem also revealed that it is possible to obtain an output signal-to-noise ratio which is greater than the input signal-to-noise ratio with the parametric amplifier in double-sideband operation.

Searching the literature on noise figure for analogous situations, one finds that confusion exists on this point of single- vs double-sideband noise figure. Uenohara[4]

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